Chapter 1: Classification of Signal and System

Houshou Chen

Dept. of Electrical Engineering,
National Chung Hsing University

E-mail: houshou@nchu.edu.tw

- Siganls:
  1. What is a signal
  2. Classification of signals
  3. Basic operations on signals
  4. Elementary signals
- Systems:
  1. What is a system
  2. Classification of systems
  3. LTI systems: circuit example
  4. More example and motivation

Signals

- Signal: a continuous-time signal $x(t)$ (discrete-time signal $x[n]$) is a function of an independent continuous variable t (discrete variable n).
- Elementary continuous-time signals:
  1. $x(t) = e^{s_0 t}, s_0 = \sigma_0 + j\omega_0$ (complex exponential)
  2. $x(t) = e^{j\omega_0 t}, s_0 = j\omega_0$ (periodic complex exponential)
  3. $x(t) = e^{\sigma_0 t}, s_0 = \sigma_0$ (real exponential)
  4. $x(t) = \cos\omega_0 t = \mathrm{Re}\{e^{j\omega_0 t}\}$ (sinusoidal signals)
  5. impulse function: $\delta(t)$
  6. unit function: $u(t)$
  7. ramp function: $r(t)$

- Elementary discrete-time signals:
  1. $x[n] = z_0^n, z_0 = r_0 e^{j\omega_0}$ (complex exponential)
  2. $x[n] = e^{j\Omega_0 n}, z_0 = e^{j\Omega_0}$ (periodic complex exponential)
  3. $x[n] = r_0^n, z_0 = r_0$ (real exponential)
  4. $x[n] = \cos\Omega_0 n = \mathrm{Re}\{e^{j\Omega_0 n}\}$ (sinusoidal signals)
  5. impulse function: $\delta[n]$
  6. unit function: $u[n]$
  7. ramp function: $r[n]$
- We will treat continuous-time and discrete-time signals separately but in parallel.

Classification of signals

1. continuous-time $x(t)$ vs. discrete-time $x[n]$
   - Usually a discrete-time signal $x[n]$ is obtained from a continuous time signal $x(t)$ by sampling:

$$x[n] = x(nT), \quad n = 0, \pm 1, \pm 2 \ldots \text{ for some fixed T.}$$

2. even vs. odd signals
   - even (real): $x(-t) = x(t)$
   - odd (real): $x(-t) = -x(t)$
   - symmetric (complex): $x(-t) = x^*(t)$
   - anti-symmetric (complex): $x(-t) = -x^*(t)$

Any signal $x(t)$ can be decompose into the even part $x_e(t)$ and the odd part $x_o(t)$ by:

$$x(t) = \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)],$$

where

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \text{ and } x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

- It is easy to check that $x_e(t) = x_e(-t)$, $x_o(t) = -x_o(t)$.

3. periodic vs. aperiodic signals

- A signal $x(t)$ ($x[n]$) is called a periodic signal if there exist real number T (integer N) such that:

$$x(t+T) = x(t) \quad (x[n+N] = x[n]).$$

- The smallest T_0 (N_0) such that :

$$x(t+T_0) = x(t) \quad (x[n+N_0) = x[n])$$

  is called the (fundamental) period of $x(t)$ ($x[n]$).
- $\frac{2\pi}{T_0}(\frac{2\pi}{N_0})$ is called the fundamental frequency ($\frac{rad}{sec}$) of $x(t)$ ($x[n]$).
- $x(t)$ ($x[n]$) is called aperiodic if it is not periodic.

4. deterministic vs. random
   - deterministic signal $x(t)$
     $\Rightarrow x(t_0)$ is a number, no uncertainity
   - random signal $x(t)$
     $\Rightarrow x(t_0)$is a random variable (with some probability specification)
     $x(t)$ = random signal = random process = stochastic process
5. energy signal vs. power signal
   - for a continuous signal $x(t)$:
     $E = \int_{-\infty}^{\infty} x^2(t)dt$: energy
     P=$\lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt$: power
     $= \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt$ if $x(t)$ is periodic with period T

- for a discrete signal $x[n]$
  $E = \sum_{n=-\infty}^{\infty} x[n]$: energy
  $P = \lim_{n\rightarrow\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} x^2[n]$: power
  $= \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$ periodic with period N
- $x(t)(x[n])$ is an energy signal
  if $0 < E < \infty$
  or is a power signal
  if $0 < P < \infty$
- A signal $x(t)$ $(x[n])$ can not be an energy signal and a power signal simultaneously.

Difference between $x(t)$ and $x[n]$

- There are many similarities between $x(t)$ and $x[n]$, but there is one important difference.
- For a continuous time $x(t) = e^{jw_0 t}$ we have:
  1. $e^{jw_1 t} \neq e^{jw_2 t}$ if $w_1 \neq w_2$, i.e., any two signals with two different frequencies are distinct.
  2. $w_1 > w_2 \Rightarrow e^{jw_1 t}$ oscillates faster than $e^{jw_2 t}$.
  3. $e^{jw_0 t}$ is periodic for any w_0, $T_0 = \frac{2\pi}{w_0}$.

- The above three properties are not true for a discrete-time signal $x[n] = e^{j\Omega_0 n}$.

1. For a discrete-time signal, we have
   $x[n] = e^{j(\Omega_0+2\pi)n} = e^{j\Omega_0 n} \times e^{j2\pi n} = e^{j\omega_o n}$
   i.e., the signal $x[n]$ at frequency $(\Omega_0 + 2\pi)$ is the same as that at frequency Ω_0, that is unlike the continuous case:
   $e^{jw_1 t} \neq e^{jw_2 t}$ if $w_1 \neq w_2$

2. I.e., for continuous-time signal, $e^{jw_0 t}$ are all distinct for distinct w_0. On the other hand, in discrete-time, the signal
   $x[n] = e^{j\Omega_0 n} = e^{j(\Omega_0+2m\pi)n}$ for any $m \in \mathbf{Z}$.
   $\Longrightarrow$ we only need to consider a frequency interval of length 2π, usually $-\pi \leq \Omega < \pi$ or $0 \leq \Omega < 2\pi$.

3. Ω_0 is larger $\Rightarrow e^{j\Omega_0 n}$ oscillate faster is not true in discrete-time case

- In discrete-time, since we only need to consider a frequency interval of length 2π, say $-\pi \leq \Omega < \pi$ or $0 \leq \Omega < 2\pi$. We have: frequencies close to 0, 2π are termed as low frequencies and frequencies close to π, or $-\pi$ are termed as high frequencies.
- I.e., As $\Omega \rightarrow 0, 2\pi$, $e^{j\Omega_0 n}$ oscillates slower, and as $\Omega \rightarrow \pi, -\pi$, $e^{j\Omega_0 n}$ oscillates faster.
- $\cos(0n) = 1, cos(\frac{\pi n}{8}), cos(\frac{\pi n}{4}), cos(\frac{\pi n}{2}), cos(\frac{\pi n}{1})$, Ω from 0 to π, $e^{j\Omega n}$ oscillates slower to faster
- $cos(\frac{3\pi n}{2}), cos(\frac{3\pi n}{4}), cos(\frac{8\pi n}{7}) cos(2\pi n) = 1$, Ω from π to 2π, $e^{j\Omega n}$ oscillates faster to slower

4. The period of discrete-time signal $e^{j\Omega_0 n}$
   - $e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} * e^{j\Omega_0 N} = e^{j\Omega_0 n}$(need $e^{j\Omega_0 N} = 1$)
     $\Rightarrow \Omega_0 N = 2\pi m \Rightarrow \frac{\Omega_0}{2\pi} = \frac{m}{N}$
     i.e., a discrete-time signal $e^{j\Omega_0 n}$ is not necessary periodic for any Ω_0. For a periodic $e^{j\Omega_0 n}$, we must have $\Omega_0 = s2\pi$, where $s \in Q$.
   - $e^{j\frac{n\pi}{4}}$ $(\Omega_o = \frac{\pi}{4} = \frac{1}{8}2\pi, N = 8)$ periodic
   - e^{j3n} $(\Omega_o = 3 \neq \frac{m}{N}2\pi)$ not periodic

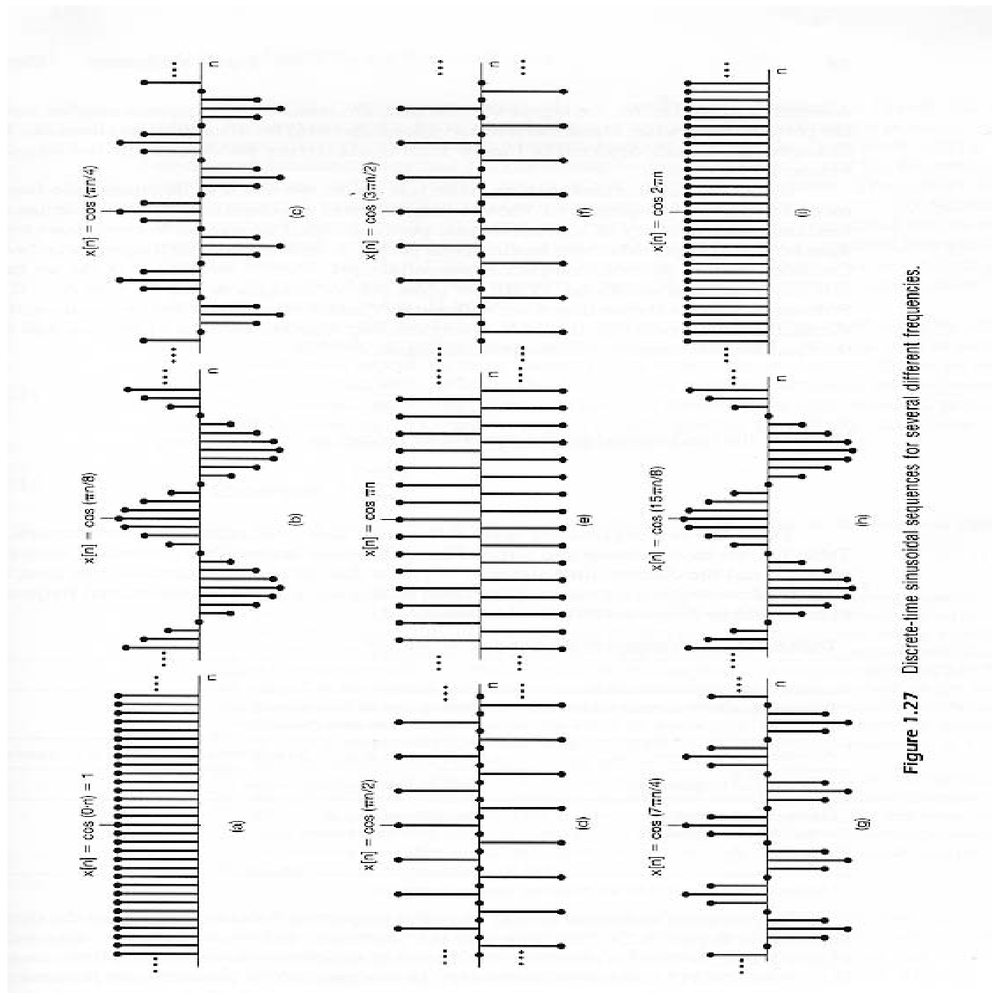
x[n] = cos (0πn) = 1
(a)
x[n] = cos (πn/8)
(b)
x[n] = cos (πn/4)
(c)
x[n] = cos (πn/2)
(d)
x[n] = cos πn
(e)
x[n] = cos (3πn/2)
(f)
x[n] = cos (7πn/4)
(g)
x[n] = cos (15πn/8)
(h)
x[n] = cos 2πn
(i)
Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Figure 1: $(-1)^n$

Operations on signals

- operation on $\begin{cases} t - axis \\ x - axis \end{cases}$ of $x(t)$

- On dependent variable $x(t)$

  i.e., given $x(t)$, $\Longrightarrow$ want to find $y(t) = Ax(t) + B$

  $$\begin{cases} y_1(t) = Ax(t) & \text{scaling first} \\ y_2(t) = y_1(t) + B & \text{shift next} \end{cases}$$

  $\Rightarrow y_2(t) = y(t) = Ax(t) + B.$

- $y(t) = Ax(t) + B$
  - Remark: $\begin{cases} |A| > 1 \quad \text{expand}(A < 0 \text{ reverse}) \\ |A| < 1 \;\; \text{compress} \\ B > 0 \;\; \text{shift up} \\ B < 0 \;\; \text{shift down} \end{cases}$

- $y(t) = 3x(t) + 4$

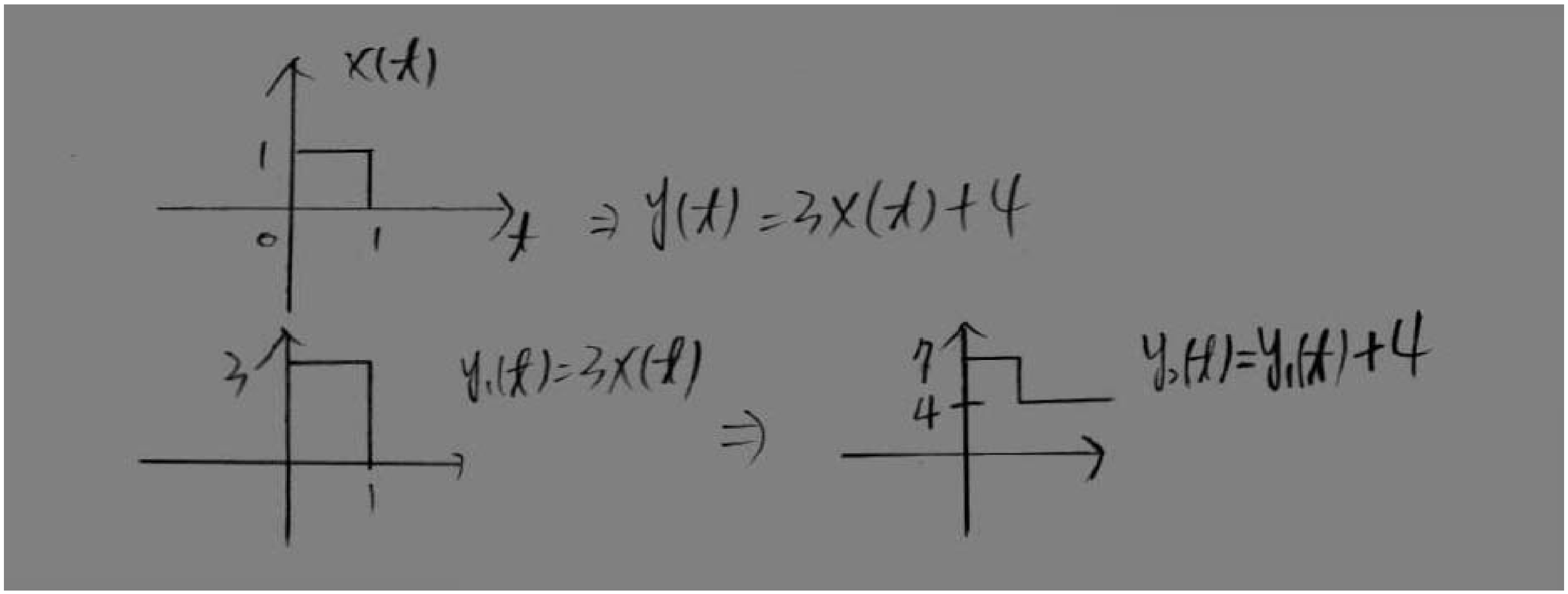

Figure 2:

If we do

$$\begin{cases} y_1(t) = x(t) + B & \text{shift next} \\ y_2(t) = Ay_1(t) & \text{scaling first} \end{cases}$$

$$\Longrightarrow y_2(t) = A(x(t) + B)$$

- Conclusion:
  - $y(t) = Ax(t) + B$ then A first $\Rightarrow$ B next
  - $y(t) = A(x(t) + B)$ then B first $\Rightarrow$ A next

- On independent variable t

i.e., given $x(t) \Rightarrow y(t) = x(at+b)$

$$\begin{cases} y_1(t) = x(t+b) & \text{shift first} \\ y_2(t) = y_1(at) & \text{scaling next} \end{cases}$$

$\Rightarrow y_2(t) = y(t) = y_1(at+b)$

- Remark: $\begin{cases} |a| > 1 & \text{compress}(a < 0 \text{ reverse}) \\ |a| < 1 & \text{expand} \\ b > 0 & \text{shift left (advance version)} \\ b < 0 & \text{shift right (delayed version)} \end{cases}$

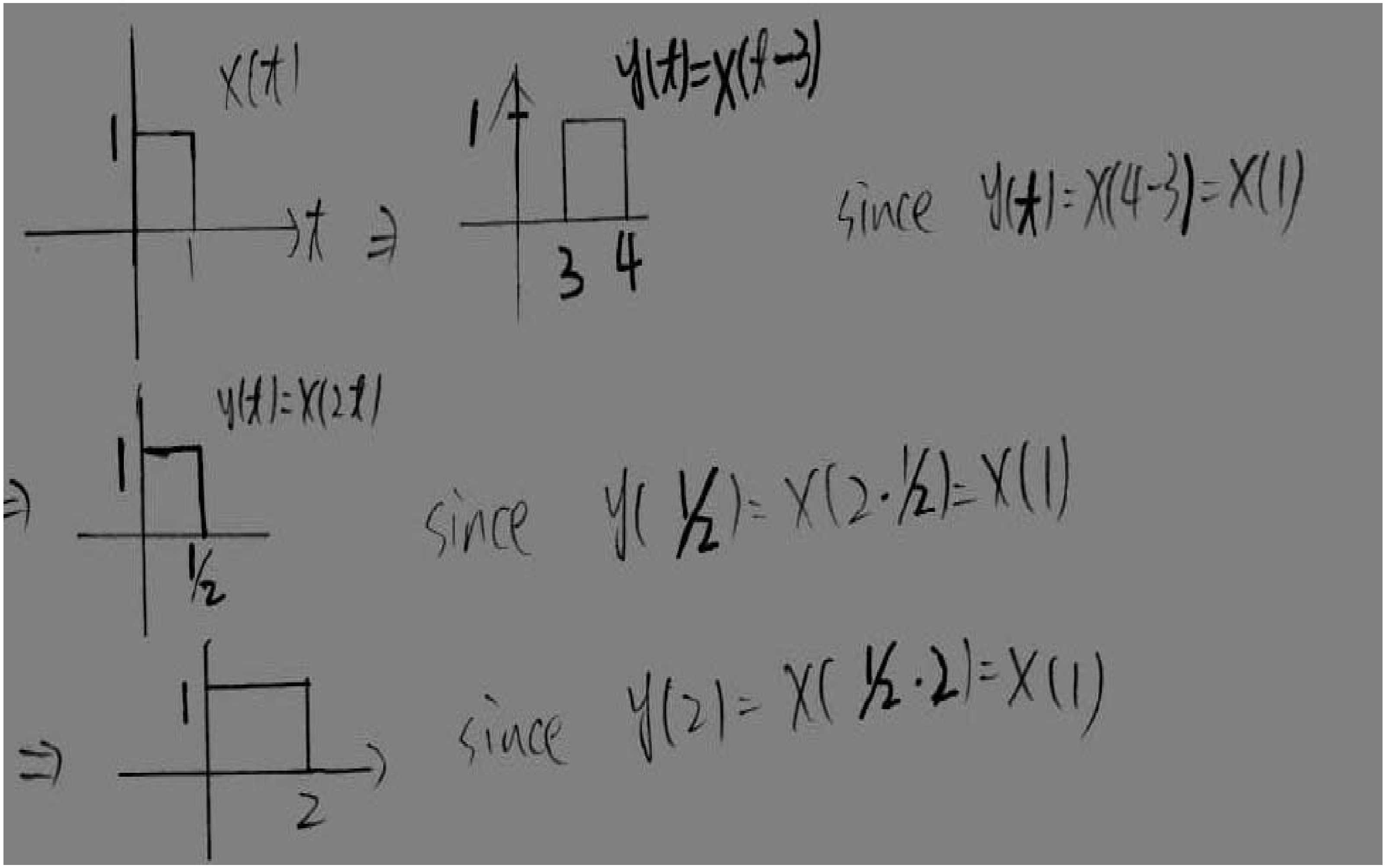

Figure 3:

$$\text{If we do} \begin{cases} y_1(t) = x(at) & \text{scaling first} \\ y_2(t) = y_1(t+b) & \text{shift next} \end{cases}$$

$$\Rightarrow y_2(t) = y_1(t+b) = x(a(t+b)) = x(at+ab)$$

Conclusion:

$$y(t) = x(at+b) \qquad b \text{ first} \Rightarrow a \text{ next}$$

$$y(t) = x(a(t+b)) \qquad a \text{ first} \Rightarrow b \text{ next}$$

Why need this: convolutional sum, integral

- $x(t) \Rightarrow Ax(t) + B$ A first,B next
- $x(t) \Rightarrow x(at + b)$ $b > 0$ shift left

  b first, a next $b < 0$ shift right

  or equivalent $x(t) \Rightarrow x(at - b)$ $b > 0$ shift right

  $x(t) \Rightarrow x(at - b)$ $b < 0$ shift left

by changing variable

$$\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \Rightarrow t - \tau = \lambda \Rightarrow \tau = t - \lambda \Rightarrow d\tau = -d\lambda$$

$$= \int_{\infty}^{-\infty} h(\lambda)x(t-\lambda)(-d\lambda) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)(d\lambda) = x(t) \bigotimes h(t)$$

$$x[n] \bigotimes h[n] = \sum_{k=\infty}^{-\infty} h[n-k]x[k]$$

$$= \sum_{m=\infty}^{-\infty} h[m]x[n-m] = x[n] \bigotimes h[n]$$

Recall $h(\tau) \Rightarrow h(t-\tau) = h(-\tau + t)(h(-\tau - (-t)) = h(-(\tau - t)))$

1. $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau) = h(t) \bigotimes x(t)$
2. $y[n] = \sum_k h[n-k]x[k] = h[n] \bigotimes x[n]$
3. $X(D)h(D)$

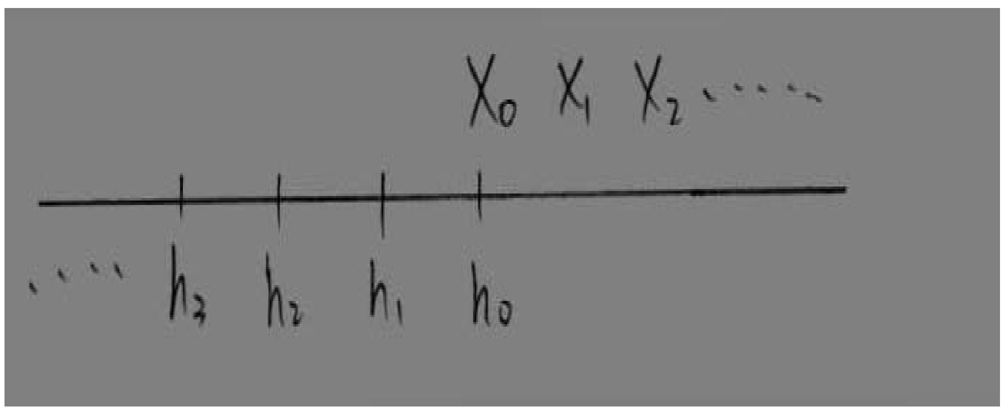

Figure 4:

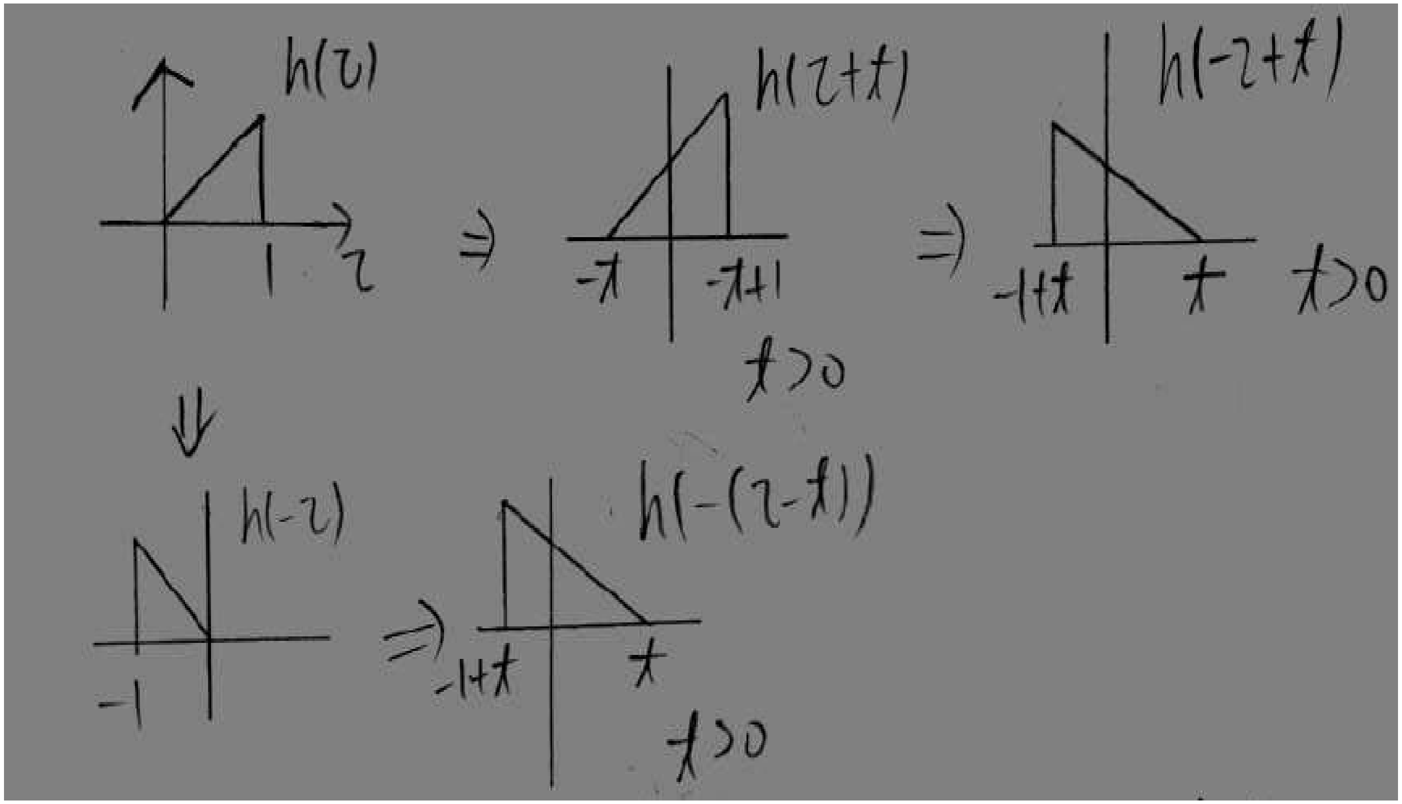

Figure 5:

Other Elementary signals

1. ramp function:

$$r(t) = \begin{cases} 0 & t \leq 0 \\ t & t \geq 0 \end{cases}$$

$$r[n] = \begin{cases} 0 & n \leq 0 \\ n & n \geq 0 \end{cases}$$

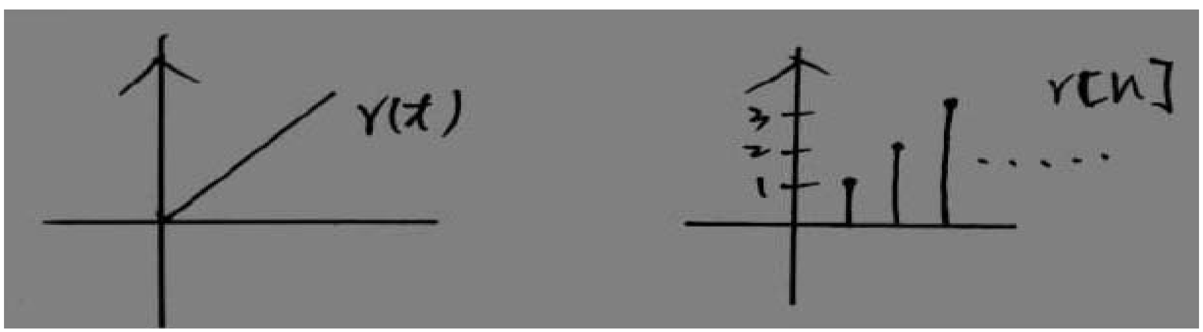

Figure 6:

2. unit function

$$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq 1 \quad \text{step function} \end{cases}$$

$$u[n] = \begin{cases} 0 & n = -1, -2, \ldots \\ 1 & n = 0, 1, \ldots \end{cases}$$

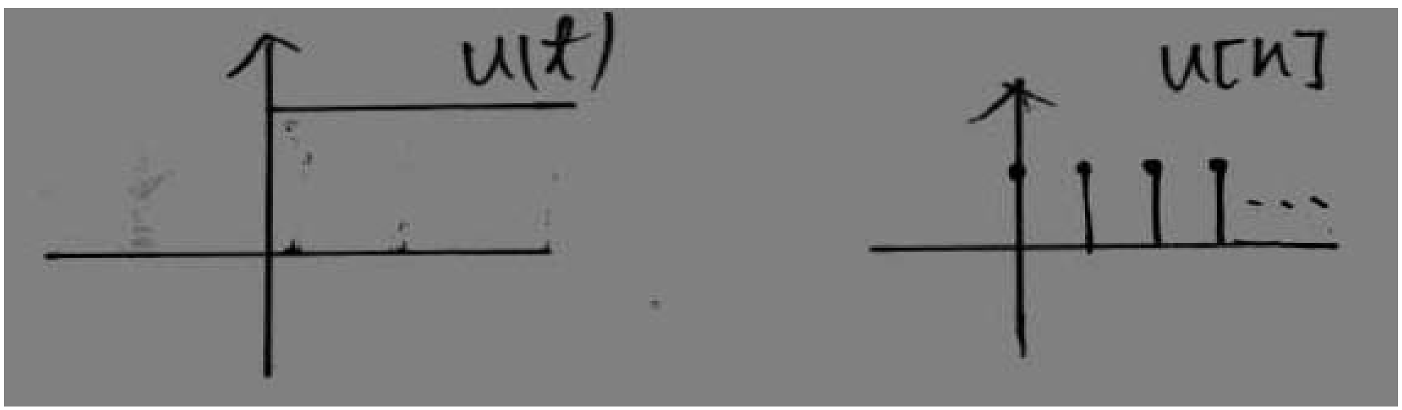

Figure 7:

Remark: Many functions $x(t)$ can be written in term of step function. This will be very useful since we can deal with the transform of $x(t)$ by the transform of $u(t)$, e.g., $r(t) = tu(t)$.

- $u(t) - u(t-1)$
- $u(t-a) - u(t-b)$

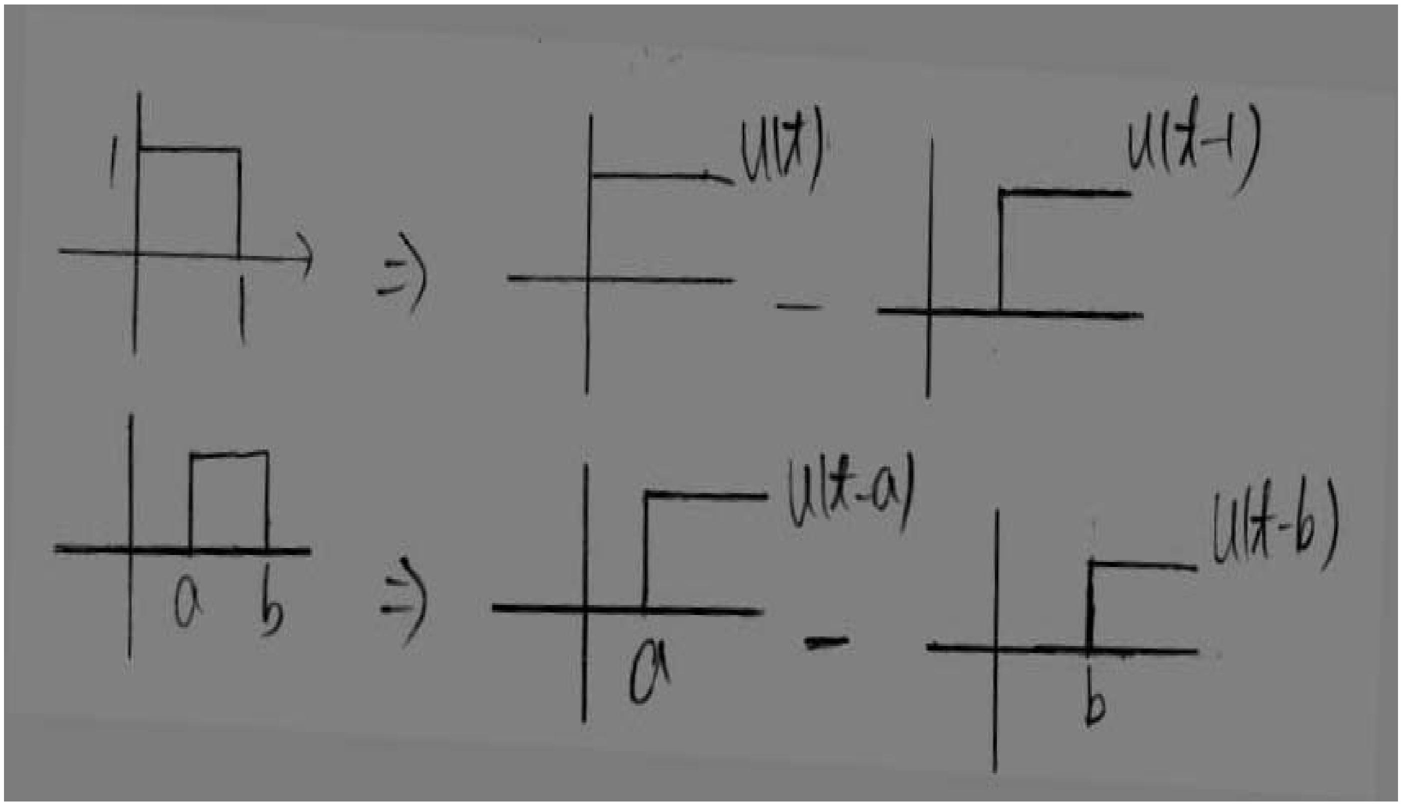

Figure 8:

- $t \cdot (u(t) - u(t-1))$
- $t \cdot (u(t) - u(t-1)) + (u(t-1) - u(t-2))$

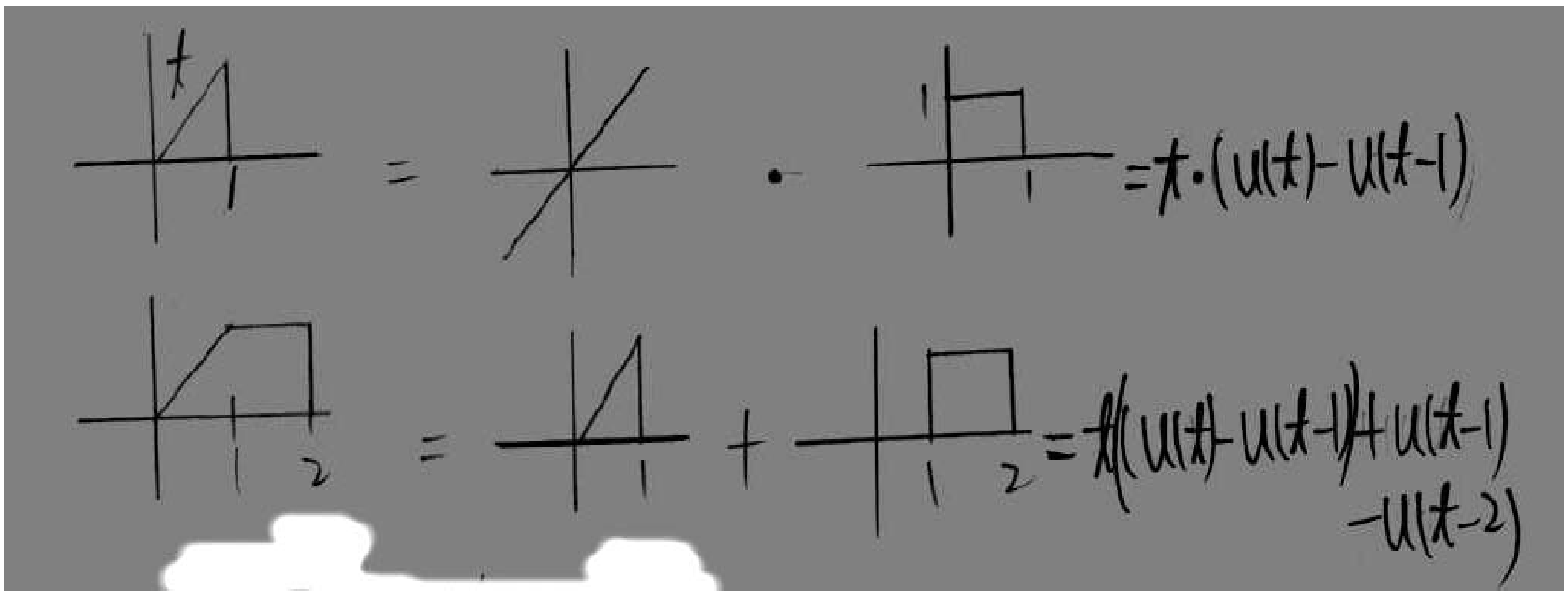

Figure 9:

In general, if we have $x(t)$ in the form as follows.

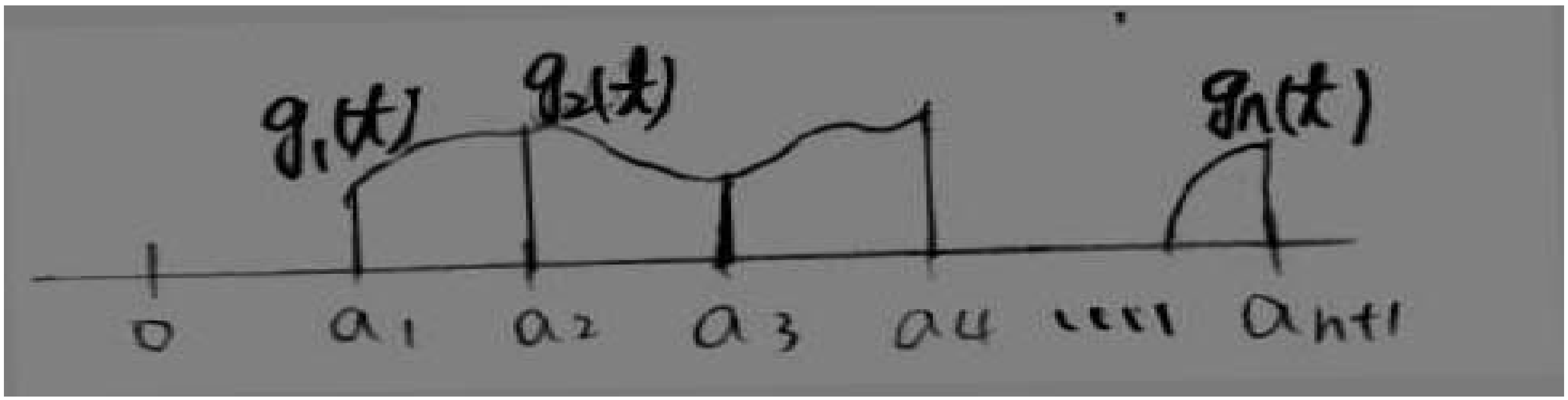

Figure 10:

We can always partition $x(t)$ into:

$$
\begin{aligned}
x(t) = g_1(t)[u(t-a_1) - u(t-a_2)] \\
&+ g_2(t)[u(t-a_2) - u(t-a_3)] \\
&+ \vdots \\
&+ g_n(t)[u(t-a_n) - u(t-a_{n+1})] \text{ as follows.}
\end{aligned}
$$

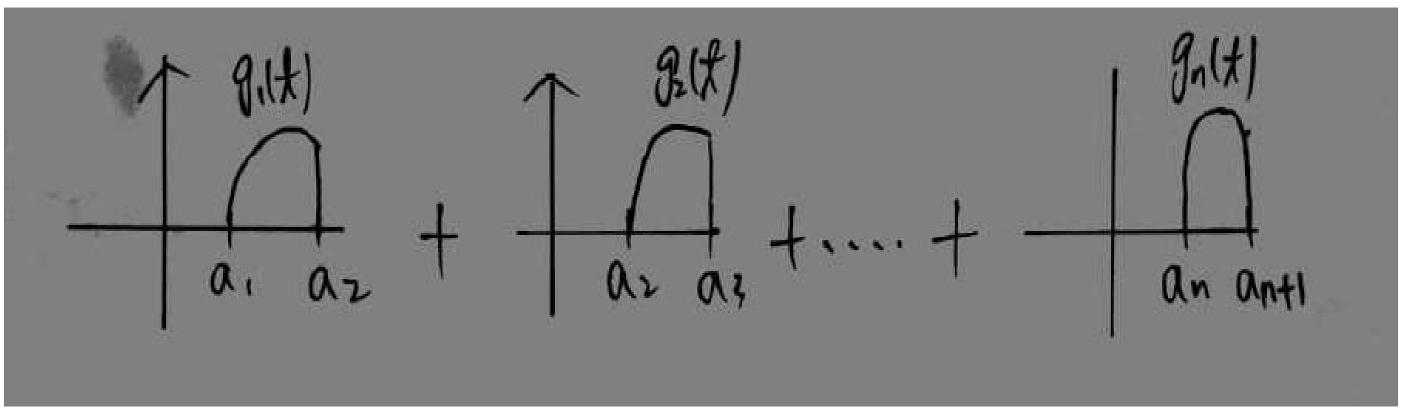

Figure 11:

3. impulse function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 \cdot \infty & t = 0 \quad \text{impulse function} \end{cases}$$

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \quad \text{delta function} \end{cases}$$

In general, $\delta(t)$ is not a function, it is a generalized function. (but $\delta[n]$ is a function).

For example, $\delta(t)$ can be defined as the limit of some function.

- We can think of the continuous-time impulse function with the property

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\text{and } \delta(t) = \begin{cases} 0 & (t \neq 0) \\ \infty & (t = 0) \end{cases}$$

- In other words, continuous-time impulse $\delta(t)$ has the property: $\delta(t) = 0$ for all t except at $t = 0$ and the total area under $\delta(t)$ is 1.

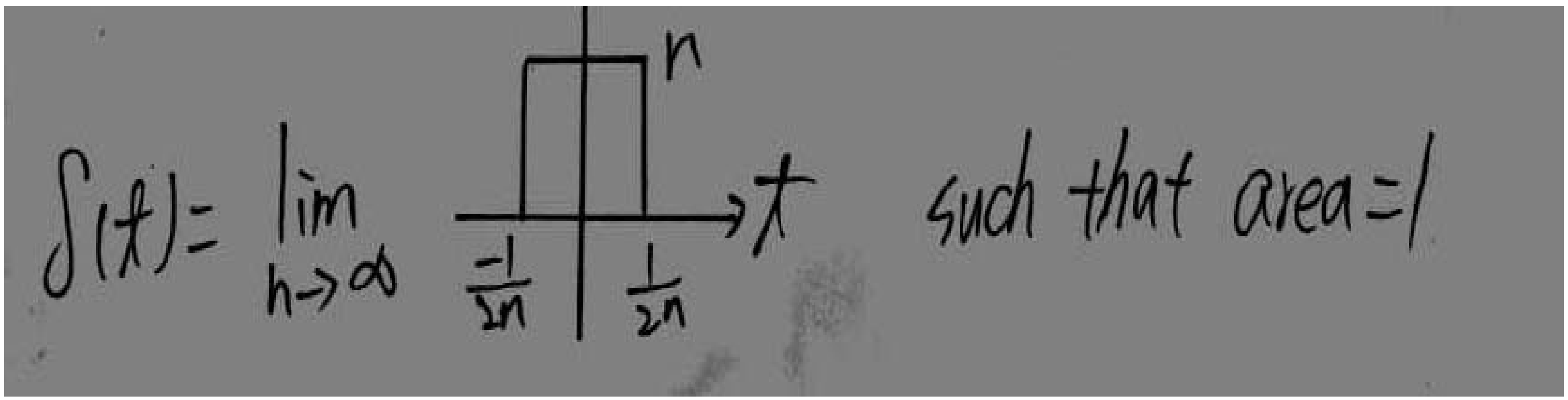

Figure 12:

Properties of impulse function

There are many property of $\delta(t)$

1. sampling property:

   $x(t) * \delta(t - t_0) = x(t_0) * \delta(t - t_0)$

2. sifting property:

   $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$

   $$\int_a^b x(t)\delta(t - t_0)dt = \begin{cases} x(t_0) & \text{if } t_0 \in [a, b] \\ 0 & \text{else} \end{cases}$$

sampling and sifting property

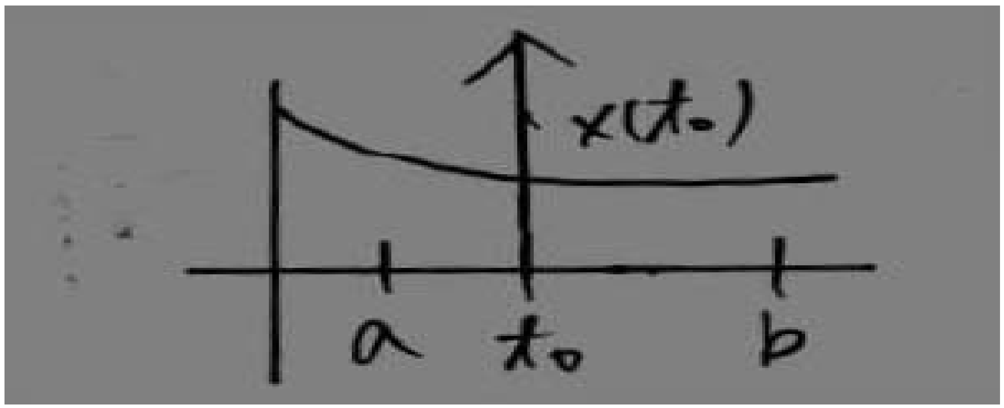

Figure 13:

3. $\delta(at) = \frac{1}{|a|}\delta(t)$

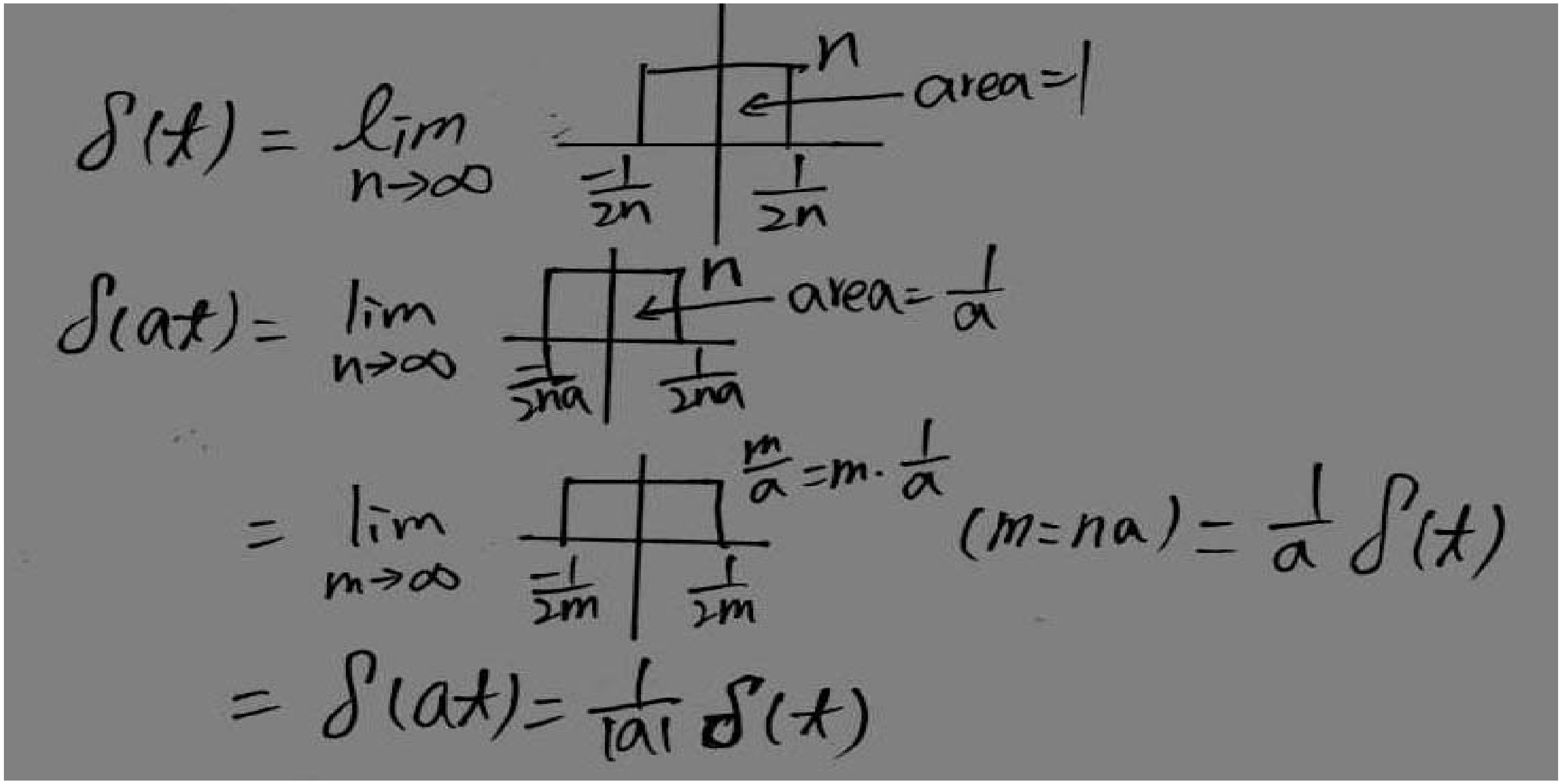

Figure 14:

4. $\delta(at+b) = \delta(a(t+\frac{b}{a})) = \frac{1}{|a|}\delta(t+\frac{b}{a})$

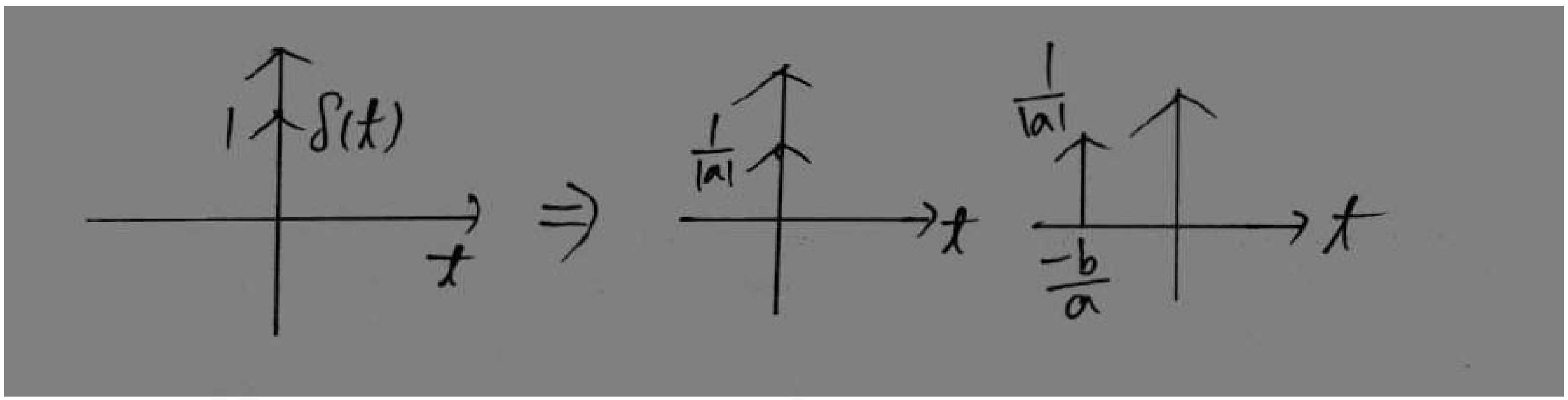

Figure 15:

- All of these properties can be proved by thinking $\delta(t)$ as a generalized function.
- From the above properties, we have

$$\begin{aligned} x(t_0) &= \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt && (\text{ by 1}) \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(\tau-t_0)d\tau && (\text{ replace } t \text{ by } \tau) \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t_0-\tau)d\tau && (\text{ by 3}) \end{aligned}$$

Since this is true for $\forall t_0 \in (-\infty, \infty)$, we can replace t_0 by t.

- Finally, we have

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t), \quad \forall t$$

$\Rightarrow x(t) = x(t) \otimes \delta(t)$

From this property, $\delta(t)$ (or $\delta[n]$) is the identity of convolutional integral (convolutional sum)

- $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

  or $= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau$ (continuous-time)

- $x[n] = \Sigma_{k=-\infty}^{\infty} x[k]\delta[n-k]$

  or $= \Sigma_{k=-\infty}^{\infty} \delta[k]x[n-k]$ (discret-time)

We see that any signal $x(t)$ ($x[n]$) can be written as the "linear combination" of $\delta(t)$ ($\delta[n]$) and it's shift version $\delta(t-\tau)$ ($\delta[n-k]$), i.e., the linear integral for continuous-time, and linear sum for discrete-time.

- Remark:

$$\begin{cases} r'(t) = u(t) \\ \\ u'(t) = \delta(t) \end{cases}$$

$$\begin{cases} \int_{-\infty}^{t} \delta(\tau) d\tau = u(t) \\ \\ \int_{-\infty}^{t} u(\tau) d\tau = r(t) \end{cases}$$

Also

$$\begin{cases} r[n] - r[n-1] = u[n] \\ \\ u[n] - u[n-1] = \delta[n] \end{cases}$$

$$\begin{cases} \Sigma_{k=-\infty}^{n} \delta[k] = u[n] \\ \\ \Sigma_{k=-\infty}^{n} u[k] = r[n] \end{cases}$$

- The relationship between $u[n]$ and $\delta[n]$
- From the identity of convolutional sum, we have

$$u(t) = \int_{-\infty}^{\infty} u(t-\tau)\delta(\tau)d\tau$$

$$= \int_{-\infty}^{t} \delta(\tau)d\tau$$

- Similarly, we have

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$$

$$= \int_{0}^{\infty} \delta(t-\tau)d\tau$$

- The relationship between $u[n]$ and $\delta[n]$
- From the identity of convolutional sum, we have

$$u[n] = \sum_{k=-\infty}^{\infty} u[n-k]\delta[k]$$

$$= \sum_{k=-\infty}^{n} \delta[k]$$

- Similarly, we have

$$u[n] = \sum_{k=\infty}^{\infty} \delta[n-k]u[k]$$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

System

A continuous-time (discrete-time) system H is an operator that transfer the input $x(t)$ $(x[n])$ into the output $y(t)$ $(y[n])$. We denote the process by

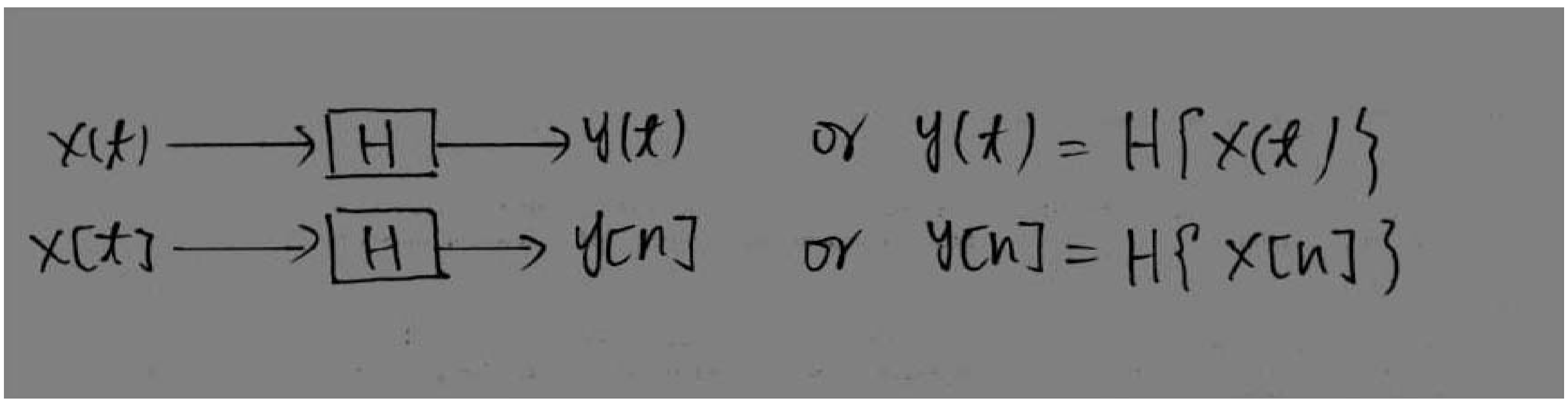

Figure 16:

Example: the RLC circuit

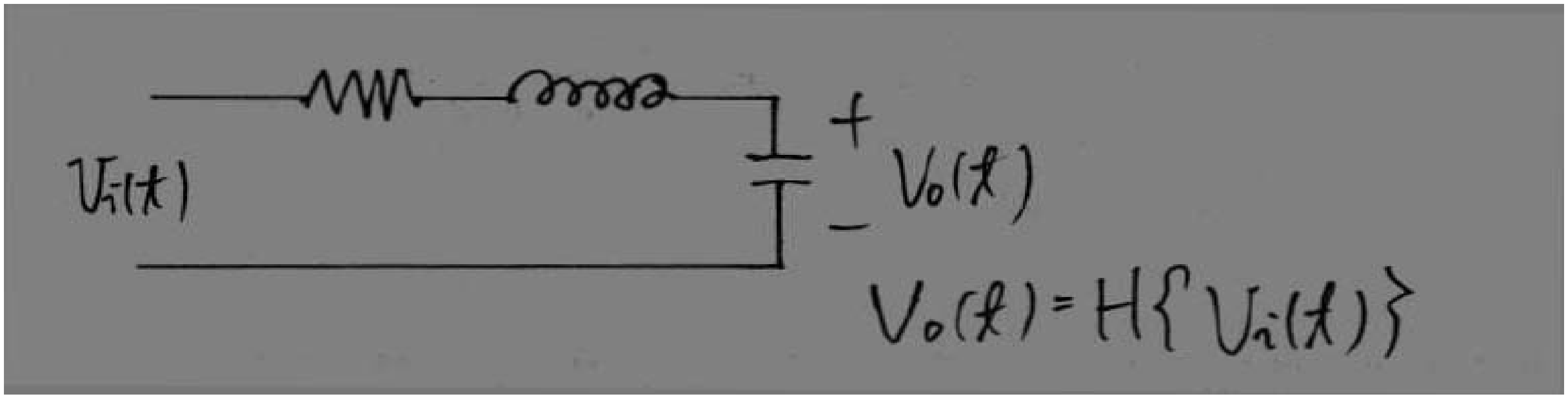

Figure 17:

How to describe the relationship between the input $v_i(t)$ and the output $v_0(t)$?

Classification of system

1. linear vs. nonlinear

   H is called linear if H has the superposition property:

$$\begin{cases} H\{x_1(t)+x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\} \\ H\{cx(t)\} = cH\{x(t)\} \end{cases}$$

$$\Leftrightarrow H\{c_1x_1(t)+c_2x_2(t)\} = c_1H\{x_1(t)\} + c_2H\{x_2(t)\}$$

$$\Leftrightarrow H\{\sum_{i=1}^{n} c_i x_i(t)\} = \sum_{i=1}^{n} c_i H\{x_i(t)\}$$

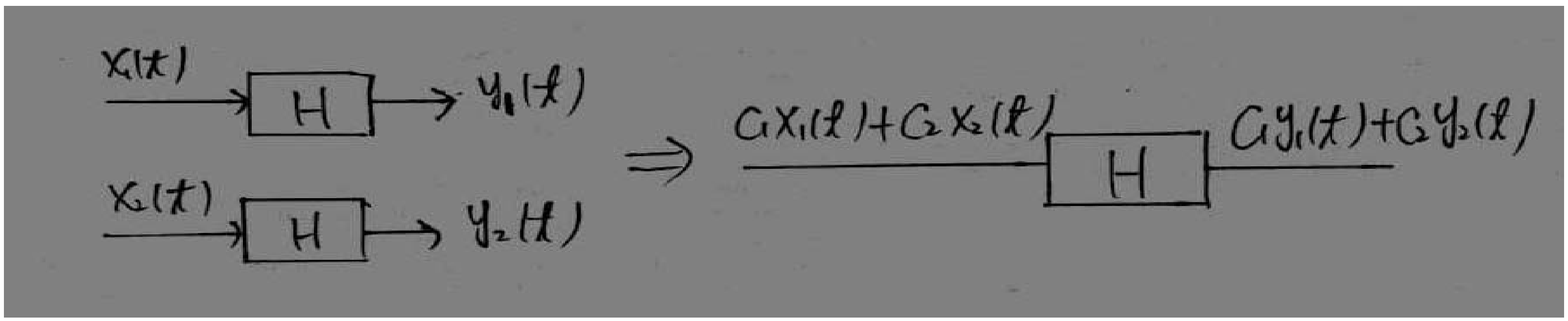

Figure 18:

2. time-invariant vs. time-variant

- H is called time-invariant if the following is true

$$H\{x(t)\} = y(t) \Longrightarrow H\{x(t-t_0)\} = y(t-t_0)$$

- I.e., a time-shift to in the input $x(t)$ results in an identical time-shift to in the output

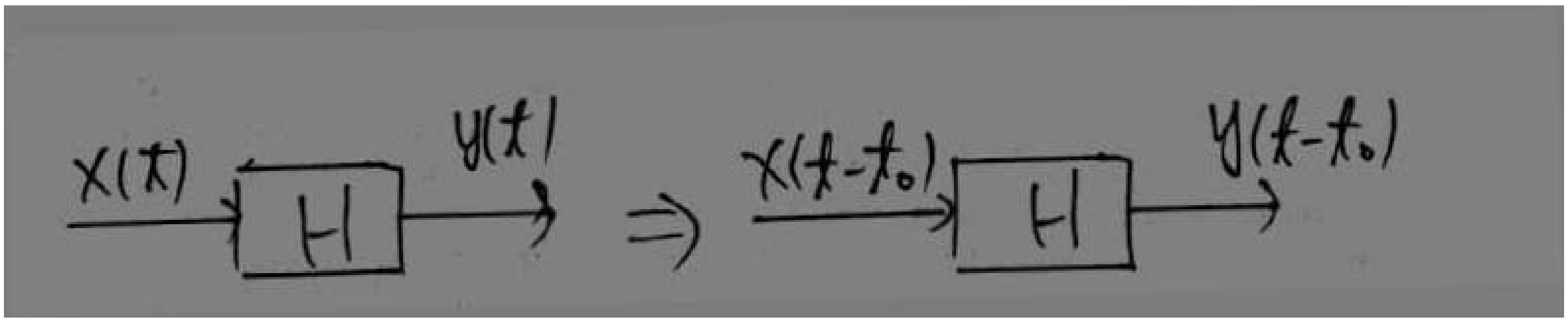

Figure 19:

3. memory vs. memoryless

- A system H is memoryless if the value $y(t_0)$ (i.e.,$y(t = t_0)$) only depends on the value $x(t_0)$ for any t_0.
- example: $y(t) = x^2(t)$ is memoryless since $y(t_0) = x^2(t_0)$ for $\forall t_0$.
- example: $y(t) = x(t-1)$ is a system with memory since $y(t_0) = x(t_0 - 1)$, e.g., $y(0) = x(-1)$. $y(t_0)$ depends on $x(t)$ at $t = t_0 - 1$, not at t_0.
- In other words, output $y(t)$ at current time $t = t_0$ is only affected by input $x(t)$ at current time $t = t_0$

4. causal vs. noncausal

- A system H is causal if the value $y(t_0)$ only depends on $\{x(t) : t \leq t_0\}$.
- I.e., current output is produce by current input and past input, not future input.
- the system $y[n] = x[n-1]$ is causal $(y[0] = x[-1])$
- the system $y[n] = x[n+1]$ is noncausal $(y[0] = x[1])$
- the system $y(t) = x(t+a)$ is causal if $a \leq 0$ and is noncausal if $a > 0$

5. stable vs. nonstable

- H is stable if $\mid x(t) \mid \leq M_x < \infty \ \forall t$
  then $\mid y(t) \mid \leq M_y < \infty \ \forall t$
- I.e., bounded input $x(t)$ produces bounded output $y(t)$

We will focus on a linear time-invariant system (LTI system) H.
If H is a LTI system, $x(t)$ and $y(t)$ are usually described by

- impulse response $h(t)$
- transfer function $H(s)$
- differential equation
- block diagram

Preview and Review: t and s domain

1. t-domain: impulse response $h(t)$

   $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

   $\Rightarrow y(t) = H\{x(t)\} = H\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\}$

   $= \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

2. s-domain: transfer function $H(s)$

   $x(t) = \int_{-\infty}^{\infty} X(w)e^{jwt}dw \Rightarrow y(t) = H\{x(t)\}$

   $= \int_{-\infty}^{\infty} X(w)H\{e^{jwt}\}dw = \int_{-\infty}^{\infty} X(w)H(w)e^{jwt}dw$

- e^{st} is an eigenfunction of a continuous-time LTL system
  $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$

  $= (\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau)e^{st} = H(s)e^{st}\ (= H(s)x(t))$

- z^n is an eigenfunction of a discrete-time LTL system
  $y[n] = \sum_{-\infty}^{\infty} h[k]x[n-k] = \sum_{-\infty}^{\infty} h[k]z^{n-k}$

  $= (\sum_{-\infty}^{\infty} h[k]z^{-k})z^n = H(z)z^n\ (= H(z)x[n])$

Change of basis: two domains

A vector x in terms of one basis $\{e_1, e_2 \cdots, e_n\}$
$x = (x_1, x_2, \cdots x_n) = x_1(100\cdots 0) + x_2(010\cdots 0) + \cdots + x_n(000\cdots 1)$
$= x_1e_1 + x_2e_2 + \cdots x_ne_n \; (\in \langle e_1, e_2 \cdots e_n\rangle)$
The same vector x in terms of another basis $\{v_1, v_2 \cdots v_n\}$

$$x = x_1e_1 + x_2e_2 + \cdots x_ne_n = x_1'v_1 + x_2'v_2 + \cdots + x_n'v_n = x'$$

- A vector x has two representations in terms of two bases $x = (x_1, x_2, \cdots, x_n) = (x_1', x_2', \cdots, x_n')$
- We can change from $\{e_i\}_{i=1}^n$ to $\{v_i\}_{i=1}^n$ and vice versa; if $\{v_i\}_{i=1}^n$ are eigenvectors, we can simplify operation $y = Ax$ in $\{e_i\}_{i=1}^n$ domain to $y = Dx$ in $\{v_i\}_{i=1}^n$ domain.

- The reason is as follows. In $\{e_i\}_{i=1}^{n}$ domain, we have $y = Ax$.
- If $Av_i = \lambda v_i$ for all i
  $\{v_1, v_2 \cdots v_n\}$ = eigenvectors with eigenvalues $\{\lambda_1, \lambda_2 \cdots \lambda_n\}$

$$x = x_1 e_1 + x_2 e_2 + \cdots x_n e_n = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_n = x'$$

$$x' = (x'_1 x'_2 \cdots x'_n) = x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_n$$

$$y = Ax = A(x'_1 v_1 + x'_2 v_2 + \cdots + x'_n v_n)$$

$$= x'_1 \lambda_1 v_1 + x'_2 \lambda_2 v_2 + \cdots + x'_n \lambda_n v_n = y'_1 v_1 + \cdots y'_n v_n$$

where $y'_i = \lambda_i x'_i$

Or equivalently,

$A(v_1 v_2 \cdots v_n) = (Av_1, Av_2, \cdots Av_n) = (\lambda_1 v_1, \lambda_2 v_2, \cdots \lambda_n v_n)$

$$= (v_1 v_2 \cdots v_n) \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \Rightarrow AV = VD \Rightarrow A = VDV^{-1}$$

$y = VDV^{-1}x \Rightarrow y' = Dx' \Rightarrow V^{-1}y = DV^{-1}x$

Motivation of LTI system

- Motivation I: O.D.E and Circuit ⇔ signal and system

A RLC circuit

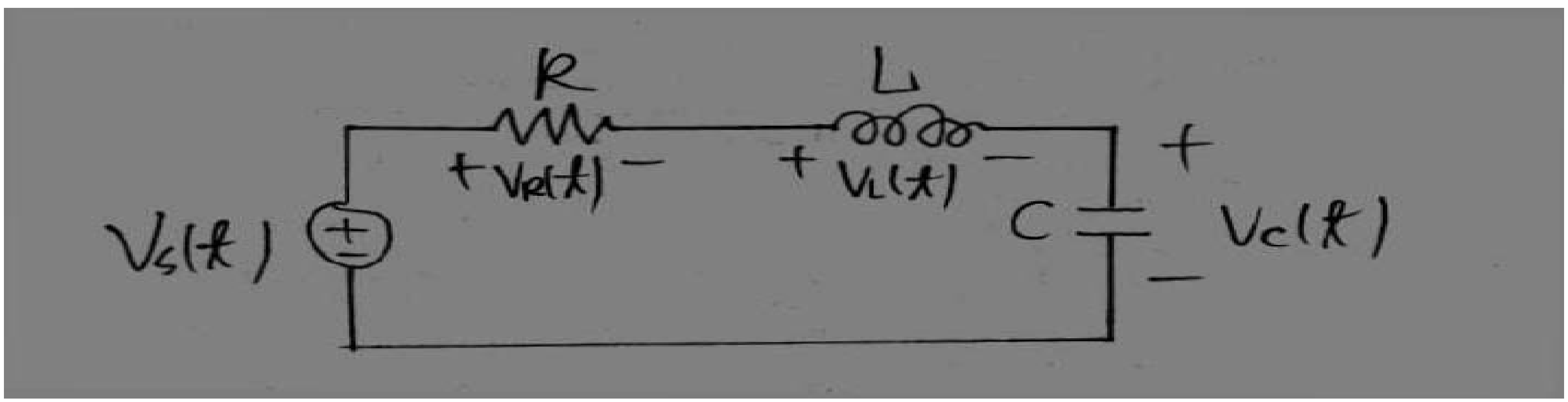

Figure 20:

Or block diagram

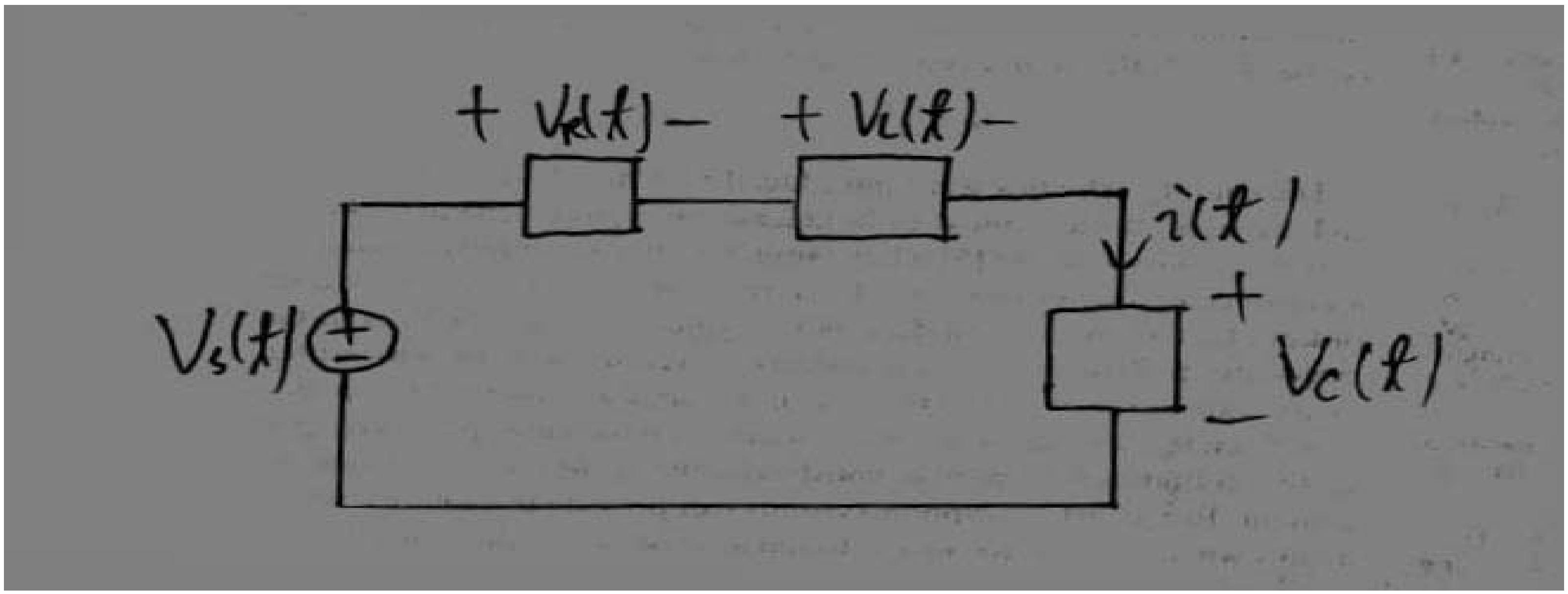

Figure 21:

From the circuit theory, we have

$$\begin{cases} V_R(t) = R \cdot i(t) \\ \\ V_L(t) = L\frac{di(t)}{dt} \\ \\ i(t) = C\frac{dV_C(t)}{dt} \Rightarrow \frac{di(t)}{dt} = C\frac{d^2V_C(t)}{dt^2} \end{cases}$$

Therefore, by KVL, we have :$V_C(t) + V_L(t) + V_R(t) = V_s(t)$

$$\Rightarrow V_R(t) = R \cdot i(t) = RC\frac{dV_C(t)}{dt}$$

$$V_L(t) = L \cdot C\frac{d^2V_C(t)}{dt^2}$$

Finally, we have

$$L \cdot C \frac{d^2 V_C(t)}{dt^2} + RC \frac{dV_C(t)}{dt} + V_C(t) = V_s(t)$$

$$\Rightarrow V_c''(t) + \frac{R}{L} V_c'(t) + \frac{1}{Lc} V_c(t) = \frac{1}{LC} V_s(t)$$

Input signal: $x(t) = V_s(t)$

output signal: $y(t) = V_c(t)$

$\Rightarrow$ The differential equation describing the relationship between input $x(t)$ & output $y(t)$ is as follows.

$$y''(t) + \frac{R}{L} y'(t) + \frac{1}{LC} y(t) = \frac{1}{LC} x(t)$$

This is a 2nd order constant coefficient linear ODE.

A complete solution $y(t)$ is given by:

$y(t) = y_h(t) + y_p(t)$ (O.D.E.)

$= y_{\text{Z.I.R}}(t) + y_{\text{Z.S.R.}}(t)$ (circuit)

$= y_{\text{natural}}(t) + y_{\text{forced}}(t)$ (circuit)

In general, y(t) for $t \geqslant t_0$ depends on both the initial state $s(t_0)$ and the input function $x(\tau)$, $t \geqslant t_0$ we write:
$y(t) = F(s(t_0); x(\tau), \tau \geqslant t_0)$
then $ZIR(t) = f(s(t_0); 0)$; $ZSR(t) = f(0; x(\tau), \tau \geqslant t_0)$

1. For a linear-system, Complete system response=ZIR+ZSR
2. We will assume $s(t_0) = 0$ from now on and turn attention to ZIR when we discuss the Laplace Transform.

1. Solving 2nd oreder O.D.E.:
   $y_h(t)$: solving $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$ (two roots λ_1 & λ_2)

   $\Rightarrow y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ ($\lambda_1 \neq \lambda_2$ distinct roots)

   or $y_h(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}$ ($\lambda_1 = \lambda_2$ repeat roots)

   or $y_n(t) = e^{\alpha_1 t}(c_1 \cos \beta_1 t + c_2 \sin \beta_1 t)$

   where $\lambda_1 = \alpha_1 + \imath\beta_1$,$\lambda_2 = \overline{\lambda_1} = \alpha_1 - \imath\beta_1$

2. Solving 1st order differential system:
   states:$i(t)$ & $v_C(t)$

   $\Rightarrow V_C'(t) = \frac{1}{c} i(t)$

   $i'(t) = \frac{1}{L} V_L(t) = \frac{1}{L}(V_s(t) - V_C(t) - V_R(t))$
   $= \frac{1}{L}(-V_C(t) - Ri(t) + V_s(t))$

$$\Rightarrow \begin{bmatrix} V_C(t) \\ i(t) \end{bmatrix}' = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} V_C(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} V_s(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + F(t)$$

in general,we have $X'(t) = Ax(t) + F(t)$
and we have the solution of the first order differential system:
$X(t) = X_h(t) + X_p(t)$ where $X_h(t)$ is obtained by
diagonalizing the matrix A

$$A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 V_1 & \lambda_2 V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\Rightarrow A = VDV^{-1}$$

$$x'(t) = VDV^{-1}x(t)$$

$$\Rightarrow \underbrace{V^{-1}x^{-1}(t)}_{Y'(t)} = D\underbrace{Vx(t)}_{Y(t)}$$

$$\Rightarrow \begin{cases} y_1(t) = c_1 e^{\lambda_1 t} \\ y_2(t) = c_2 e^{\lambda_2 t} \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_1(t) & v_2(t) \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Summary

From the above example, we can see that there are several ways to describe the relationship between the input $x(t)$ and the output $y(t)$ for a LIT sytem $x(t) \leftrightarrow y(t)$. These are:

1. Block diagram

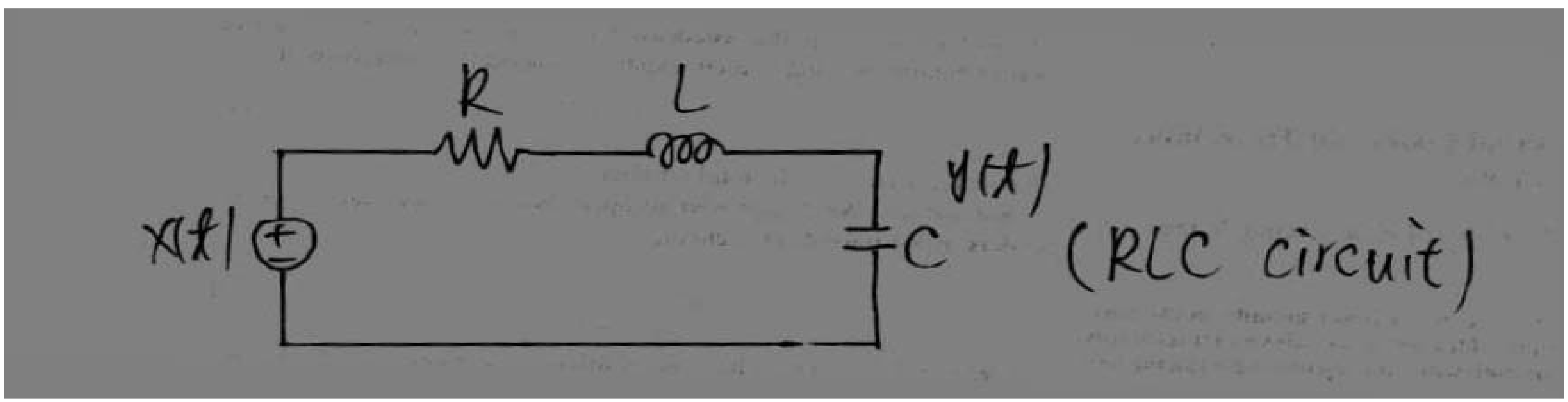

Figure 22:

2. differential equation (SISO system)

$y'' + \frac{R}{L}y'(t) + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$ ($\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$ two roots)

$\Rightarrow y(t) = y_h(t) + y_p(t) = y_{ZIR}(t) + y_{ZSR}(t)$

where

$$y_h(t) = \begin{cases} c_1 e^{\lambda t} + c_2 e^{\lambda t} & (\lambda_1 \neq \lambda_2 \text{ real}) \\ c_1 e^{\lambda t} + c_2 t e^{\lambda t} & (\lambda_1 = \lambda_2 \text{ real}) \\ y_h(t) = c_1 e^{\alpha_1 t}(\cos\beta t + \sin\beta t) & (\lambda_1 = \lambda_2 = \alpha + \imath\beta) \end{cases}$$

3. differential system (MIMO system)

$$y'(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}' = A \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + F(t)$$

$$\Rightarrow y(t) = y_h(t) + y_p(t)$$

$$\& \; y_h(t) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

where $Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2 (\lambda_1 \neq \lambda_2)$

4. our focus

$$\begin{cases} \text{time domain } h(t)\text{: impulse response} \\ \text{frequency domain } H(s)\text{: transfer function} \end{cases}$$

We can find the transfer function $H(s)$, or the frequency response $H(jw)$ ($H(e^{j\Omega})$) directly from the circuit diagram or from the differential equation (system). After that, we can get the impulse response $h(t)$ from $H(s)$. The idea is connecting with phasors in circuit theory.

Phasors

R: $V_R(t) = Ri(t)$

$i(t) = e^{jwt} \Rightarrow V_R(t) = \overbrace{R}^{Z_R} e^{jwt} \Rightarrow Z_R = R(\text{ independce})$

L: $V_L(t) = L\frac{di(t)}{dt}$

$i(t) = e^{jwt} \Rightarrow V_L(t) = \overbrace{L \cdot jw}^{Z_L} e^{jwt} \Rightarrow Z_L = jwL \; (SL)$

C: $i(t) = C\frac{dV_C(t)}{dt}$

$V_C(t) = e^{jwt} \Rightarrow i(t) = \overbrace{jwC}^{Z_C} e^{jwt} \Rightarrow Z_C = \frac{1}{jwC} \;\; (\frac{1}{sC})$

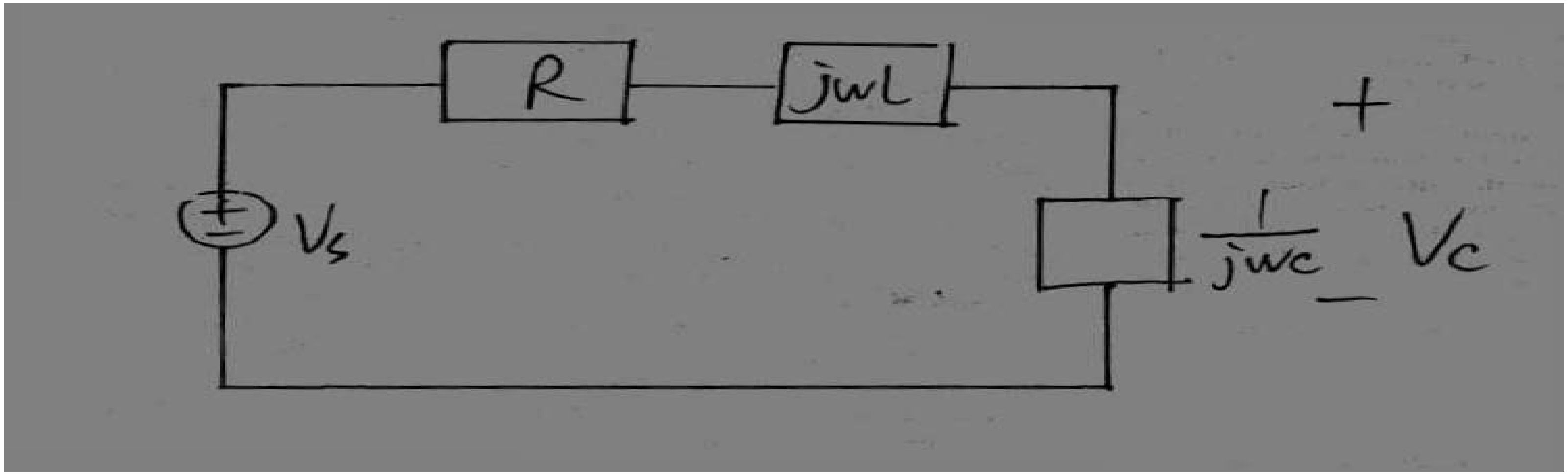

Figure 23:

We can replace R by R, C by $1/jwC$, and L by jwL; then by KVL or KCL we can solve the transfer function $H(s)$ directly from the circuit diagram.

therefore by voltage divider, we have

$$V_c = \underbrace{\frac{\frac{1}{jwc}}{\frac{1}{jwc} + R + jwL}}_{H(jw)} V_s, \ (*jw\frac{1}{L} \text{ on top and bottom})$$

i.e. $H(jw) = \frac{\frac{1}{LC}}{(jw)^2 + \frac{R}{L}jw + \frac{1}{LC}}$

or $H(s) = \frac{\frac{1}{LC}}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$

Note: O.D.E. $V_c''(t) + \frac{R}{L}V_c'(t)\frac{1}{LC} + V_c(t) = \frac{1}{LC}V_s(t)$

- It seems that we can find $H(s)$ aslo from the ODE.

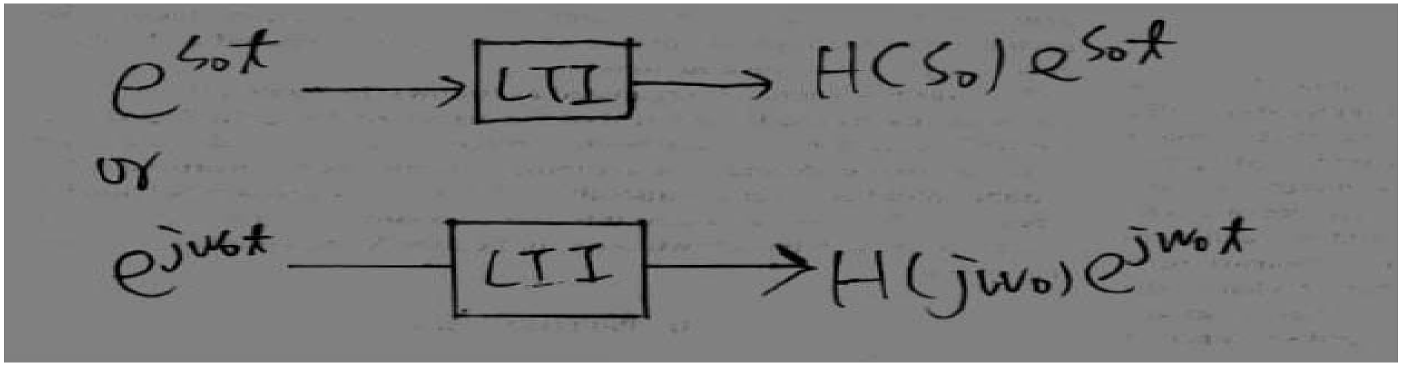

Figure 24:

- $x(t) = e^{jwt}$ (or in general $x(t) = e^{st}$) is an eigenfunction of a continuous-time LTI system.
- $x[n] = e^{j\Omega n}$ (or in general $x[n] = z^n$) is an eigenfunction of a discrete-time LTI system.
- Let $x(t) = e^{jwt}$ then $y(t) = H(jw)e^{jwt}$
- Let $x[n] = e^{j\Omega n}$ then $y[n] = H(e^{j\Omega n})e^{j\omega n}$

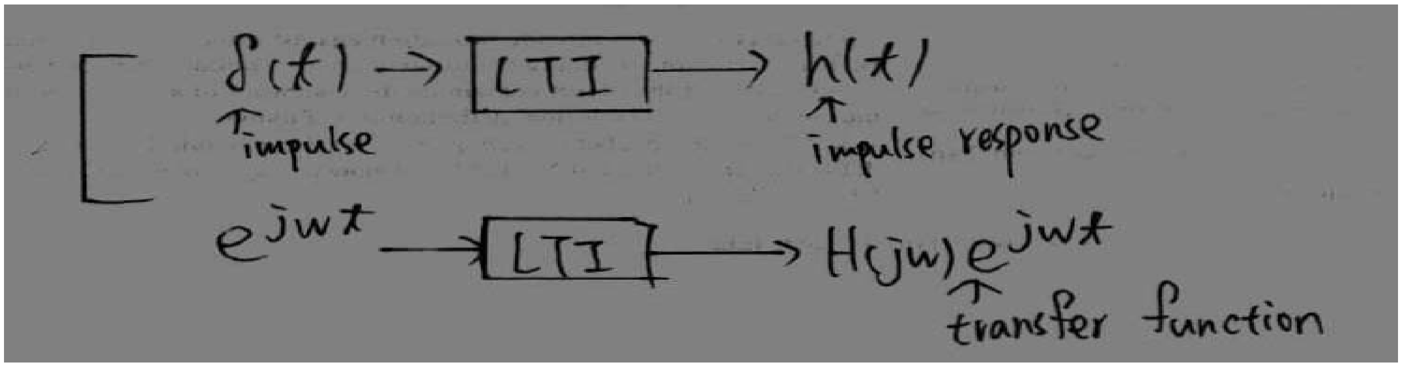

Figure 25:

I.e., mathematically, for a LTI system H, we have

1. $h(t) = H\{\delta(t)\}$, $h[n] = H\{\delta[n]\}$
2. $H(jw) = \frac{H\{e^{jwt}\}}{e^{jwt}}$, $H(e^{j\Omega}) = \frac{H\{e^{j\Omega n}\}}{e^{j\Omega n}}$

e.g. the ODE for RLC circuit is:

Let $x(t) = e^{jwt}$, then $y(t) = H(jw)e^{jwt}$ $y''(t) + \frac{R}{L}y' + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$

then $y'(t) = (jw)H(jw)e^{jwt}$, $y''(t) = (jw)^2 H(jw)e^{jwt}$

$\Rightarrow ((jw)^2 + \frac{R}{L}jw + \frac{1}{RL})H(jw)e^{jwt} = \frac{1}{RL}e^{jwt}$

$\Rightarrow H(jw) = \frac{\frac{1}{LC}}{(jw)^2 + \frac{R}{L}jw + \frac{1}{LC}}$

This is always true for any nth order linear constant coefficient ODE. That is, given a differential equation for a LTI system

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$

$$= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_1 x^1(t) + b_0 x(t)$$

i.e., $\sum_{i=1}^{n} a_i y^{(i)}(t) = \sum_{j=1}^{m} b_j x^{(j)}(t)$

Substitute: $x(t) = e^{jwt}$ & $y(t) = H(jw)e^{jwt}$

into the ODE & use the fact $\frac{d^i}{dt^i}e^{jw} = (jwt)^i e^{jwt}$ we have

$$(a_n(jw)^n + a_{n-1}(jw)^{n-1} + \cdots + a_1(jw) + a_0)H(jw)e^{jwt}$$

$$= (b_m(jw)^m + b_{m-1}(jw)^{m-1} + \cdots + b_1(jw) + b_0)e^{jwt}$$

$$\leftrightarrow H(jw) = \frac{b_m(jw)^m + \cdots + b_1(jw) + b_0}{a_n(jw)^n + \cdots + a_1(jw) + a_0}$$

$$H(s) = \frac{b_m s^m + \cdots + b_1 s^m + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

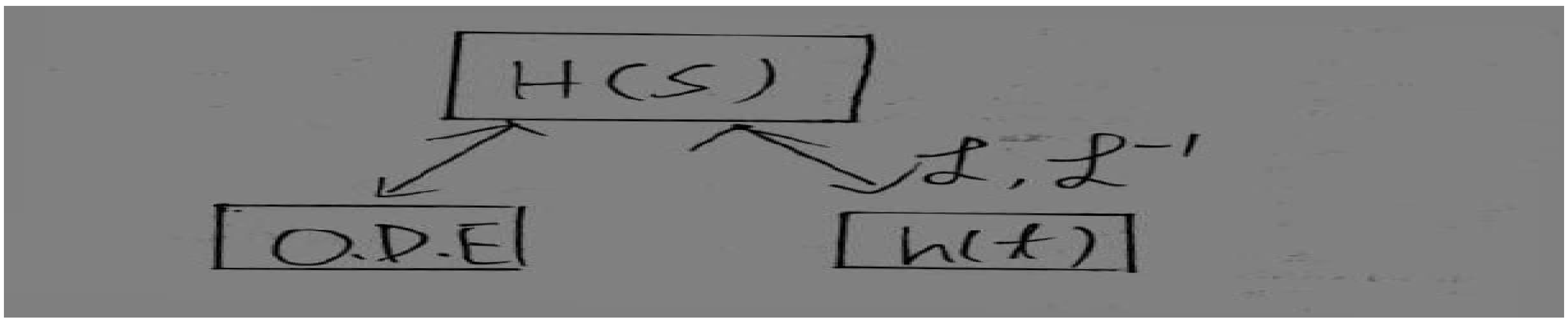

Figure 26:

Usually $H(s) = \frac{N(s)}{D(s)}$ $(degD(s) = n)$

$= \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \cdots + \frac{A_n}{s+p_n}$ (assume D(s) has n distinct toots)

(by P.E.F. partial fraction Expansion)

$\Rightarrow h(t) = L^{-1}\{H(s)\}$

$\Rightarrow h(t) = A_1 e^{-p_1 t} u(t) + A_2 e^{-p_2 t} u(t) + \cdots + A_n e^{-p_n t} u(t)$

In general,

block diagram $>$ differential system
$>$ O.D.E
$>$ h(t)(H(S))

where $>$ means providing more information.

In signal & system,we study the zero-state response

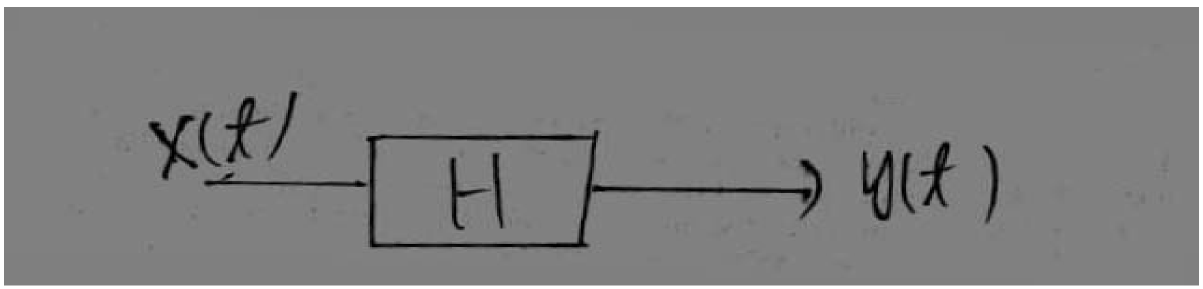

Figure 27:

in particular,the system H will be a L.I.T. system.(linear & time invariant)

Motivation II: (linear algebra$\Leftrightarrow$ signal & system)

$$A = \begin{bmatrix} a & d & c & b \\ b & a & c & d \\ c & b & a & d \\ d & c & b & a \end{bmatrix} \text{ (circulant matrix)}$$

- How to find the eigenvectors and eigenvalues for the circulant matrix A?
- We can use the fact that A represents a discrete-time LTI system to find the eigenvectors and eigenvalues.

The matrix A represents a LTI system for a discrete-time with periodic input $x[n]$. That is, if x is a periodic input, then $y = Ax$ is the periodic output with the fact that $y[n]$ is obtained by the circular convolution between $x[n]$ and $h[n]$:

$$y[n] = \sum_{k=1}^{N} x[k]h[n-k]$$

In this example, we have $h[n] = (a, b, c, d)$.

Find the eigenvalues and eigenvectors for A. First,we can find eigenvalues of A by

$$Av = \lambda v \Rightarrow (\lambda I - A)v = 0$$

with $v \neq 0$

Therefore we must have $\lambda I - A$ is a singular matrix, i.e. $\det(\lambda I - A) = 0$ (characteristic polynomial).

This is a poly of degree n if A is a $n \times n$ matrix.

In general, it is not easy to find the eigenvalues for a given $n \times n$ matrix A.

For this circulant matrix,we can show that

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = (e^{j\frac{2\pi}{4}\cdot 0\cdot n})_{n=0,1,2,3} = (i^0)_{n=0,1,2,3}$$

$$v_2 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = (e^{j\frac{2\pi}{4}\cdot 1\cdot n})_{0\le n\le 3} = (i^{1\cdot n})_{0\le n\le 3}$$

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \left(e^{j\frac{2\pi}{4}\cdot 2\cdot n}\right)_{0\le n\le 3} = \left(i^{2\cdot n}\right)_{0\le n\le 3}$$

$$v_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = \left(e^{j\frac{2\pi}{4}\cdot 3\cdot n}\right)_{0\le n\le 3} = \left(i^{3\cdot n}\right)_{0\le n\le 3}$$

are eigenvectors of A.

eigenvalue of v_1

$$= a + d + c + b$$

eigenvalue of v_2

$$(a - c) + i(d - b)$$

eigenvalue of v_3

$$(a + c) - (d + b)$$

eigenvalue of v_4

$$(a - c) - i(d - b)$$

Moveover, v_1, v_2, v_3, v_4 are orthogonal vectors, i.e.,

$$(v_i, v_j) = 0 \text{ for any } i \neq j$$

.

Let $e_i = \frac{1}{\sqrt{4}} v_i \Rightarrow \{e_1, e_2, e_3, e_4\}$ are orthonormal eigenvector for A.

In other words,we have $A \underbrace{[e_1, e_2, e_3, e_4]}_{V} = \underbrace{[e_1, e_2, e_3, e_4]}_{V} D$

where $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{bmatrix}$,

and λ_i is an eigenvalue of e_i.

we can define $a = h(0), b = h(1), c = h(2), d = h(3)$ then

$$A = \begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} = \begin{bmatrix} h((n-k))_4 \end{bmatrix}$$

where $h(3)_4 = h(-1)$, $h(2)_4 = h(-2), \cdots$

Then

$$Ax = \sum_{k=0}^{3} h[n-k]x[k] \qquad = \sum_{k=0}^{3} h[k]x[n-k]$$

- This is just the discrete-time convolution sum.
- If we let $x[n] = e^{j\frac{2\pi}{4}nk_0}$ $(k_0 = 0, 1, 2, 3)$

$$\Rightarrow Ax = \sum_{k=0}^{3} h[k]e^{j\frac{2\pi}{4}(n-k)k_0}$$

$$= \underbrace{\sum_{k=0}^{3} h[k]e^{-j\frac{2\pi}{4}kk_0}}_{\lambda} \cdot \underbrace{e^{j\frac{2\pi}{4}nk_0}}_{x[n]}.$$

- i.e., $e^{j\frac{2\pi}{4}nk_0}$, $(0 \leq k_0 \leq 3)$ is an eigenvector of A with eigenvalue $\sum_k h[k]e^{-j\frac{2\pi}{4}kk_0}$.

In matrix language, we have

$$\Rightarrow y = Ax \quad \text{-time domain}$$

$$y = VDV^{-1}x \text{ (since } V^{-1} = V^t)$$

$$= VDV^Tx$$

$$\Rightarrow V^Ty = DV^Tx$$

$$\Rightarrow y' = Dx' \quad \text{-frequency domain}$$

- If V is an orthonormal matrix,then $V^T = V^{-1}$

In general,we can show

$$A = \begin{bmatrix} h(0) & h(N-1) & h(1) \\ h(1) & h(0) & h(2) \\ h(2) & h(1) & h(3) \\ \vdots & & \vdots \\ h(N-1) & h(N-2) & h(0) \end{bmatrix}$$

always has eigenvectors

$$\frac{1}{\sqrt{N}}(e^{j\frac{2\pi}{N}\cdot 0n})_{0\le n\le N-1} = e_1$$
$$\frac{1}{\sqrt{N}}(e^{j\frac{2\pi}{N}\cdot 1n})_{0\le n\le N-1} = e_2$$
$$\vdots$$
$$\frac{1}{\sqrt{N}}(e^{j\frac{2\pi}{N}\cdot (N-1)n})_{0\le n\le N-1} = e_N, \text{ (N eigenvectors)}$$

Such $\{e_1, \ldots, e_N\}$ are orthonormal eigenvector for A and

$$Ax = \sum_k h[n-k]x[k] = \sum_k h[k]x[n-k]$$

Similarly, if we let $x[n] = e^{j\frac{2\pi}{N}nk_0}$, $0 \leq k_0 \leq N-1$

$$\Rightarrow Ax = \sum_k h[k]e^{j\frac{2\pi}{N}(n-k)k_0}$$

$$= \underbrace{\sum_k h[k]e^{j\frac{2\pi}{N}kk_0}}_{\lambda} \cdot \underbrace{e^{j\frac{2\pi}{N}nk_0}}_{x[n]}$$

Also $y = Ax = VDV^{-1}x \Rightarrow V^{-1}y = DV^{-1}x$

$$\Rightarrow y = DV^T x$$

$$\Rightarrow y' = Dx'$$